

# Multivariate Reliability Modelling with Empirical Bayes Inference

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## Abstract

Recent developments in technology permit detailed descriptions of system performance to be collected and stored. Consequently, more data are available about the occurrence, or non-occurrence, of events across a range of classes through time. Typically this implies that reliability analysis has more information about the exposure history of a system within different classes of events. For highly reliable systems, there may be relatively few failure events. Thus there is a need to develop statistical inference to support reliability estimation when there is a low ratio of failures relative to event classes. In this paper we aim to show how empirical Bayes methods can be used to estimate a multivariate reliability function for a system by modelling the vector of times to realise each failure root cause.

## 1. Introduction

The motivation for this research is based on experience of modelling system reliability in collaboration with the UK aerospace industry. The focus of our work has focussed upon reliability assessment within new product development. A modelling framework has been developed to provide decision support about reliability decisions during system design and development (Walls, Quigley and Marshall, 2006).

Our premise is that the initial design specification will be informed by information from the performance of heritage systems. This is the case for the evolutionary design processes common in the aerospace industry. For example, changes to an existing design may be in response to a weakness experienced in an earlier generation of the system, or may be motivated by the need to develop improved functionality through, for example, technological innovation. For an evolutionary design process, the demand for change may constitute a mix of reactive and proactive motivations and hence reliability modelling will be informed by information from the in-service histories of operational systems as well as knowledge about the likely impact of innovation on the new design. During the development phase, additional decisions will be made to change the design in terms of, for example, component and materials selection, board layout, manufacturing process, and maintenance policy. Thus the model needs to capture the engineering design knowledge as well as relevant event history data about potential faults, often referred to as concerns, within the new design that may result in reliability problems in service if not removed or the effects mitigated.

A key characteristic of our model is the de-coupling of the engineering concern, which may or may not be realised as a fault, and the conditional distribution of the time until realisation of the concern assuming it to be a fault within the design. The former is inferred from expert elicitation processes to obtain prior probability distributions, while the latter is inferred from data for heritage systems that are similar in design or expected environmental exposure. Within the modelling process we map each engineering concern to a root cause, which describes the characteristics of the failure should it occur. As such we require a probability

distribution for each class of root cause. Typically there are many classes and few events. Fully Bayesian approaches to such a problem would require engineers to assess, not only the likelihood of an identified concern being realised, but also the time until it will be realised in operational age. Our experience has shown the elicitation of such times to be problematic, with engineering experts being vague or uncomfortable making such assessments. This is partly due to the way in-service performance data is fed back to the design engineer.

Here we explore an empirical based solution to this problem. An empirical Bayes based methodology provides a sound basis for constructing conditional distribution functions for each root cause. Broadly, this method initially assesses a distribution for the time to failure for all causes by pooling all data and subsequently assessing a covariance structure between root causes to permit adjustments from the pool for each specific root cause. This leads to each root cause having a unique distribution function. The covariance structure is obtained by constructing what could be considered empirical prior distributions, which are updated in the usual Bayesian manner to adapt to the specific root cause.

Specifically, we use a multinomial distribution to capture the sampling variability, where for each root cause is partitioned into time into intervals with each interval assigned a parameter to measure the likelihood that should such a fault exist within a design then it would be realised in that time interval. The set of probabilities for any root cause are constrained to lie within a simplex. The Dirichlet distribution is regarded as a convenient generic prior for the vector of probabilities within each root cause. Moreover, we assume that the vectors of probabilities across root causes are independent and identically distributed from a Dirichlet distribution. Thus the number of events realised for a specific root cause are conditionally independent in relation to another root cause. Taking the expectation of the multinomial distribution with respect to the Dirichlet measure provides a probability measure for each root cause which is independent and identically distributed. Using this distributional form a likelihood function is constructed from which parameter estimates and confidence intervals can be constructed for the Dirichlet prior distribution. For each root cause a posterior distribution is obtained by updating the empirically estimated prior in the usual Bayesian manner.

An illustrative example based on an industrial case is described. We explore the proposed method for constructing the reliability functions for each root cause and combining this with the expert judgement describing the engineering concerns. We discuss appropriate reliability statistics and demonstrate the usual decision support. Finally we reflect on the proposed methodology by examining issues concerning the classification of failures and the impact on assessing the system reliability and we discuss the problems concerning the double counting of data with respect to point and interval estimates. We consider the proposal from a practical perspective, with reference to cognitive limitations of experts in providing fully specified prior distributions and the need for empirically based solutions, at least in part.

## **2. The Model**

The application of Empirical Bayes (EB) within the context of risk or reliability is not new: Martz and Waller (1991) discuss the technique generally; Vesely et al (1994) discuss an application to emergency diesel generators for binary data; Vaurio (2002, 2005) use EB for estimating the rate of common cause failures; Ferdous et al (1995) use EB to support inference for the Weibull distribution within a software reliability growth context; Grabski and Sarhan (1996) combine spline density estimates for prior distributions with Empirical Bayes for inference with the exponential distribution; Vaurio and Jankala (2006) use EB

within a Poisson modelling framework; Sohn (1999) makes use of the methodology with response surface modelling of categorical quality characteristics of possible designs; Quigley et al (2007) use it to model the rate of occurrence of railway accidents; and Bedford et al (2006) use it to estimate probability within a fault tree model.

We develop a model for the time to failure of an item, where we assume that an item fails due to the realisation of an engineering concern. Should a concern be realised, it is considered a fault. Each concern is classified a priori into a mutually exclusive and exhaustive set of root causes. The operational times experienced until realisation of a fault are assumed to be statistically independent. The operational times until realisation of a fault within a root cause class are assumed identically distributed.

We denote the number of root cause classes by  $J$ . The operational time to realisation of faults is partitioned into  $I$  mutually exclusive and exhaustive partitions. It is assumed that there are  $N_j$  faults in the design associated with root cause class  $j$ . We seek a prior distribution on the  $I \times J$  matrix, denoted by  $\mathbf{P}$ , whose  $(i,j)$  element is the probability that a fault associated with root cause class  $j$  will be realised in time period  $i$ , which is denoted by  $p_{ij}$ .

Therefore, we can express the probability that an item will not fail by time  $t_0$  conditioned on the matrix  $\mathbf{P}$  and the vector  $\tilde{N} = (N_1, \dots, N_J)$  as:

$$P\left(T_s > t_0 \mid \tilde{N}, \mathbf{P}\right) = \prod_{j=1}^J \left(1 - \sum_{i=1}^{t_0} p_{ij}\right)^{N_j} \quad (1)$$

Inference for the model is supported through historical data analysis on similar items and through expert engineering judgment. The expert judgement is to construct prior distributions for the vector  $\tilde{N}$  through eliciting engineering concerns and assessing the likelihood each will be realised as a fault in operation.

### 3. Inference

We assume failure event data are available from similar designs. Denote the number of faults that were realised in time period  $i$  for root cause  $j$  as  $m_{ij}$  and denote  $\mathbf{M}$  as the corresponding matrix of data. We obtain the following likelihood function for root cause  $j$ , which is a function of the vector  $\tilde{P}_j = (p_{1j}, \dots, p_{Ij})$  :

$$L_j\left(\tilde{P}_j\right) = \binom{\sum_{i=1}^I m_{ij}}{m_{ij}} \prod_{i=1}^I p_{ij}^{m_{ij}} \quad (2)$$

It is assumed that the multinomial distribution can be used to represent the number of times to first realise a fault within a design that will be classified as root cause  $j$ .

The Empirical Bayes (EB) methodology will be applied by assuming that prior to witnessing any data, the prior distribution for the vector  $\tilde{P}_j$  is exchangeable for all  $j$ . Specifically, we assume the prior distribution to be the following Dirichlet distribution:

$$\pi(p_1, \dots, p_l) = \frac{\Gamma\left(\sum_{i=1}^l a_i\right)}{\prod_{i=1}^l \Gamma(a_i)} \prod_{i=1}^l p_i^{a_i-1}, \quad p_i \geq 0, \sum_{i=1}^l p_i = 1, a_i > 0 \quad (3)$$

We apply the Dirichlet prior distribution (3) and take the expectation of  $\tilde{P}_j$  with (2) for the  $j^{th}$  root cause and obtain a new Likelihood function which is a function of the parameters in the prior distribution:

$$L_j(a_1, \dots, a_l) \propto \frac{\Gamma\left(\sum_{i=1}^l a_i\right)}{\prod_{i=1}^l \Gamma(a_i)} \frac{\prod_{i=1}^l \Gamma(a_i + m_{ij})}{\Gamma\left(\sum_{i=1}^l (a_i + m_{ij})\right)}, \quad a_i > 0 \quad (4)$$

As the distribution of the number of faults exposed by root cause is exchangeable, the likelihood function for the data becomes the following:

$$L(a_1, \dots, a_l) = \prod_{j=1}^J \frac{\Gamma\left(\sum_{i=1}^l a_i\right)}{\prod_{i=1}^l \Gamma(a_i)} \frac{\prod_{i=1}^l \Gamma(a_i + m_{ij})}{\Gamma\left(\sum_{i=1}^l (a_i + m_{ij})\right)}, \quad a_i > 0 \quad (5)$$

We then seek the Maximum Likelihood Estimate (ML) of  $a_i$  for all  $i$  using (5) which we denote by  $\hat{a}_i$ .

### 3.1 Posterior and Predictive Distribution

Since the posterior distribution is unique for each root cause class we use the subscript  $j$ . However since each class belongs to the Dirichlet family of distributions, we substitute the estimates  $\hat{a}_i$  into the posterior to obtain the following:

$$\pi_j(p_1, \dots, p_l | \mathbf{M}) = \frac{\Gamma\left(\sum_{i=1}^l (\hat{a}_i + m_{ij})\right)}{\prod_{i=1}^l \Gamma(\hat{a}_i + m_{ij})} \prod_{i=1}^l p_i^{\hat{a}_i + m_{ij} - 1}, \quad p_i \geq 0, \sum_{i=1}^l p_i = 1, \hat{a}_i > 0 \quad (6)$$

The model describing the aleatory uncertainty within any root cause class is the multinomial distribution. Taking the expectation of a generic multinomial distribution with respect to the posterior distribution (6) to obtain a predictive distribution for the  $j^{th}$  root cause gives:

$$\Pr(N_{1j} = n_{1j}, \dots, N_{lj} = n_{lj} | N_j = n_{\cdot j}) = \binom{n_{\cdot j}}{n_{1j} \dots n_{lj}} \frac{\Gamma\left(\sum_{i=1}^I \left(\hat{a}_i + m_{ij}\right)\right)}{\Gamma\left(\sum_{i=1}^I \left(\hat{a}_i + m_{ij} + n_{ij}\right)\right)} \prod_{i=1}^I \frac{\Gamma\left(\hat{a}_i + m_{ij} + n_{ij}\right)}{\Gamma\left(\hat{a}_i + m_{ij}\right)} \quad (7)$$

This can be represented as:

$$\Pr(N_{1j} = n_{1j}, \dots, N_{lj} = n_{lj} | N_j = n_{\cdot j}) = \binom{n_{\cdot j}}{n_{1j} \dots n_{lj}} \frac{\prod_{i=1}^I \prod_{k=1}^{n_{ij}} \left(\hat{a}_i + m_{ij} + n_{ij} - k\right)}{\prod_{k=1}^{n_{\cdot j}} \left(\sum_{i=1}^I \left(\hat{a}_i + m_{ij} + n_{ij}\right) - k\right)}, \quad \hat{a}_i > 0 \quad (8)$$

### 3.2 Combining Expert Judgement

Assume the prior distribution describing the number of faults within each root cause, denoted by  $\Pr(N_j = n_{\cdot j})$  has been fully specified by an expert. Taking the expectation of (7) with respect to this prior gives a predictive distribution unconditional of the number of faults within the design. Care must be taken as there are restriction on the number of realisations within any time interval as they are constrained to sum to  $n_{\cdot j}$ . We evaluate the probability that the item fails for the first time due to a fault within root cause class  $j$  after time  $t_0$  assuming three different parametric forms: the Binomial distribution as an example of low dispersion; the Poisson for medium dispersion; and the Negative Binomial for large dispersion. To evaluate the probability the item fails after time  $t_0$  we multiply the probabilities for each root cause class. Note that as  $t_0$  tends to infinity for each of these classes, the probability of item survival tends to the probability that no faults are in the design. To present a succinct closed form expression of the probability that the item fails for the first time due to a fault within root cause class  $j$  after time  $t_0$  we provide first order approximations.

#### *Binomial Prior Distribution*

Assume the expert has specified a Binomial prior distribution for the number of faults that will be realised as root cause  $j$ . We express this prior as:

$$P(N_j = n_j) = \binom{k}{n_j} q_j^{n_j} (1 - q_j)^{k - n_j}, \quad q_j > 0, n_j = 0, 1, \dots, k \quad (9)$$

Using (1) we consider the expectation with respect to  $N_j$  to obtain the probability conditional on the parameters  $\mathbf{P}$ :

$$\begin{aligned}
P(T_j > t_0 | \mathbf{P}) &= E_{N_j} \left[ \left( 1 - \sum_{i=1}^{t_0} p_{ij} \right)^{N_j} \right] \\
&= \left( 1 - q_j \sum_{i=1}^{t_0} p_{ij} \right)^k
\end{aligned} \tag{10}$$

From previous results, it is known that the posterior distribution for  $P_j$  is has a Dirichlet distribution. We are interested in the distribution of the convolution of  $p_{ij}$  which has a Beta distribution. Taking the expectation of (10) results in the following:

$$\begin{aligned}
P(T_j > t_0 | \mathbf{M}) &= E_{\sum_{i=1}^{t_0} p_{ij}} \left[ \left( 1 - q_j \sum_{i=1}^{t_0} p_{ij} \right)^k \right] \\
&= \sum_{r=0}^k \binom{k}{r} (-q_j)^r E_{\sum_{i=1}^{t_0} p_{ij}} \left[ \left( \sum_{i=1}^{t_0} p_{ij} \right)^r \right] \\
&= \sum_{r=0}^k \binom{k}{r} (-q_j)^r \prod_{y=0}^{r-1} \left[ \frac{y + \sum_{i=1}^{t_0} \hat{a}_i + m_{ij}}{y + \sum_{i=1}^I \hat{a}_i + m_{ij}} \right]
\end{aligned}$$

This expression can be approximated by substituting the mean of the convolution of the probabilities directly into the expression for the expectation to give:

$$P(T_j > t_0 | \mathbf{M}) \approx \left( 1 - q_j \frac{\sum_{i=1}^{t_0} \hat{a}_i + m_{ij}}{\sum_{i=1}^I \hat{a}_i + m_{ij}} \right)^k \tag{11}$$

The formula in (11) has an intuitive appeal. The ratio  $\sum_{i=1}^{t_0} \hat{a}_i + m_{ij} / \sum_{i=1}^I \hat{a}_i + m_{ij}$  is the MLE of the probability a fault will be realised before or during time  $t_0$  and  $q_j$  is the probability a concern will be realised as a fault. Thus the product is the probability a concern will be realised as a fault within the first  $t_0$  time periods. As there are  $k$  concerns assumed within a design, the expression provides an estimate of the probability that all  $k$  concerns are realised after time  $t_0$ .

### *Poisson Prior Distribution*

We consider evaluating the probability the item fails for the first time due to a root cause  $j$  fault after time  $t_0$  assuming the expert has provided a Poisson prior distribution with mean  $\lambda_j$  given  $\mathbf{P}$ :

$$\begin{aligned}
P(T_j > t_0 | \mathbf{P}) &= E_{N_j} \left[ \left( 1 - \sum_{i=1}^{t_0} p_{ij} \right)^{N_j} \right] \\
&= e^{-\lambda_j \sum_{i=1}^{t_0} p_{ij}}
\end{aligned} \tag{12}$$

As with the Binomial example, we approximate the expectation of (12) by substituting the mean of the convolution of  $p_{ij}$ 's:

$$\begin{aligned}
P(T_j > t_0 | \mathbf{M}) &= E_{\sum_{i=1}^{t_0} p_{ij}} \left[ e^{-\lambda_j \sum_{i=1}^{t_0} p_{ij}} \right] \\
&\approx e^{-\lambda_j \frac{\sum_{i=1}^{t_0} \hat{a}_i + m_{ij}}{\sum_{i=1}^I \hat{a}_i + m_{ij}}}
\end{aligned} \tag{13}$$

The probability of a fault being realised within the first  $t_0$  time periods is expressed by the exponent of the exponential function, i.e.  $\sum_{i=1}^{t_0} \hat{a}_i + m_{ij} / \sum_{i=1}^I \hat{a}_i + m_{ij}$  and it is multiplied by the expected number of faults within the design providing the expected number of faults realised within the first  $t_0$  time periods. As such, the resulting formula is quite intuitive.

### Negative Binomial Distribution

The Negative Binomial distribution can be obtained through mixing a Poisson distribution with a Gamma distribution (Greenwood and Yule (1920) as cited in Johnson et al (1993)), as the following demonstrates:

$$\begin{aligned}
P(N_j = n_j) &= \int_0^\infty \frac{\lambda_j^{n_j} e^{-\lambda_j}}{n_j!} \frac{\beta^\alpha \lambda_j^{\alpha-1} e^{-\beta \lambda_j}}{\Gamma(\alpha)} d\lambda_j \\
&= \frac{\Gamma(n_j + \alpha)}{\Gamma(\alpha) n_j!} \left( \frac{\beta}{\beta + 1} \right)^\alpha \left( \frac{1}{\beta + 1} \right)^{n_j}, \quad \alpha > 0, \beta > 0, n_j = 0, 1, 2, \dots
\end{aligned} \tag{14}$$

To obtain an approximation of the probability the item will not fail within the first  $t_0$  time periods due to root cause  $j$ , we treat  $\lambda_j$  in (13) as though it were a Gamma random variable and take the expectation.

$$\begin{aligned}
P(T_j > t_0 | \mathbf{M}) &= E_{\lambda_j} \left[ e^{-\lambda_j \frac{\sum_{i=1}^{t_0} \sum_{j=1}^n a_i + m_{ij}}{\sum_{i=1}^n a_i + m_{ij}}} \right] \\
&= \left( \frac{\beta}{\beta + \frac{\sum_{i=1}^{t_0} \sum_{j=1}^n a_i + m_{ij}}{\sum_{i=1}^n a_i + m_{ij}}} \right)^\alpha
\end{aligned} \tag{15}$$

#### 4. Illustrative Example

This example aims to show how the proposed methods can be used to estimate the reliability of a new design, which is a variant of an existing item. The reliability statistic of interest is the duration of the failure free operating time. The existing design had 171 faults exposed during operation. These have been classified into 8 different root causes. The data have been obtained from a fleet of 200 items and the time to the first occurrence of a fault has been extracted for analysis. There have been a considerable number of modifications on the old design to produce the new design. An extensive elicitation exercise has been conducted on the new design, whereby several engineering concerns have been identified and assessed for likelihood of being realised as a fault in operation and an associated root cause class identified should a failure be realised. Three of the eight root cause classes have been used for concerns. Time has been partitioned into five intervals. First we consider the prior and posterior estimates and then consider the predictive distribution for the failure free operating time of the design.

##### 4.1 Prior and Posterior Distribution

The MLE's of the parameters have been solved using equation (5) and are given in Table 1.

Table 1 MLE of parameters for prior distribution

<i>Parameter</i>	<i>MLE</i>
$a_1$	5.7908
$a_2$	2.1397
$a_3$	7.4878
$a_4$	3.8746
$a_5$	0.5099



Table 2 provides a summary of both the EB prior estimates of the probability that a fault will be realised in the fleet within each of the time intervals and the empirical estimate for each of the root causes classes.

Table 2 Comparison of empirical estimates and Empirical Bayes estimates

<i>Time Period</i>	<i>Empirical Bayes Prior Estimates</i>	<i>Root Cause 1</i>	<i>Root Cause 2</i>	<i>Root Cause 3</i>	<i>Root Cause 4</i>	<i>Root Cause 5</i>	<i>Root Cause 6</i>	<i>Root Cause 7</i>	<i>Root Cause 8</i>
1	0.29	0.34	0.20	0.00	0.30	0.50	1.00	0.33	0.18
2	0.11	0.09	0.00	1.00	0.07	0.00	0.00	0.00	0.08
3	0.38	0.27	0.20	0.00	0.50	0.50	0.00	0.33	0.49
4	0.20	0.10	0.60	0.00	0.13	0.00	0.00	0.33	0.25
5	0.03	0.19	0.00	0.00	0.00	0.00	0.00	0.00	0.00

As the engineering concerns that were elicited were mapped to only three root cause classes (1, 4 and 8), we shall develop the posteriors for these classes only. Figure 1 illustrates the means of the posterior distributions for the probability of realising a fault within each of the time intervals. The prior probabilities are included for comparison. Figure 1 shows that this approach to inference does not impose a monotonic function on the rate of occurrence of failures but allows natural characteristics to be revealed through the data, such as the mode in time period 3.

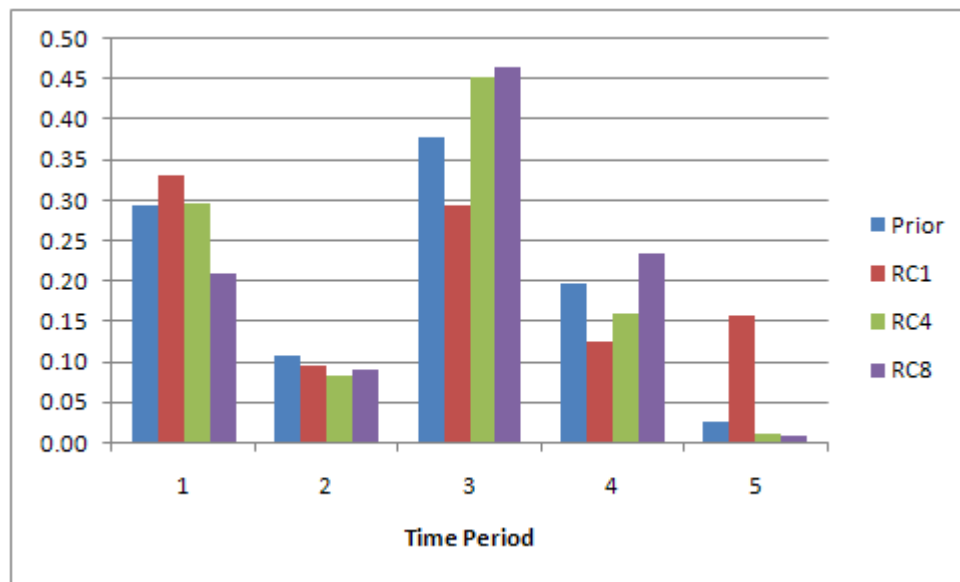


Figure 1 Posterior and prior probabilities for realising a fault within the fleet within each time interval for each root cause class

#### 4.2 Predictive Distribution – Failure Free Operating Time

We seek inference on the duration of failure free operating time for the item. We have prior distribution for each of the three root cause classes. During the elicitation process a Poisson

distribution was agreed for each of the prior distributions. For comparison and sensitivity analysis we will also consider the Negative Binomial and the Binomial distributions. Note that for this paper we use the approximations presented earlier for convenience.

#### *Poisson Prior*

Table 3 summaries the probability that no item in a fleet of 200 will fail due to each of the root causes by the end of each of the time periods. Table 3 shows that root cause 8 is less of an issue that the concerns associated with root cause 1 and 2, which have very similar profiles in comparison.

Table 3 Probability no item in the fleet will fail due given each root cause by end of time period.

<i>Time Period</i>	<i>Root Cause 1</i>	<i>Root Cause 4</i>	<i>Root Cause 8</i>
1	0.0540	0.0411	0.4934
2	0.0237	0.0168	0.3653
3	0.0018	0.0001	0.0754
4	0.0006	0.00002	0.0341
5	0.0002	0.00002	0.0334

As the new item is not being used across a fleet of 200 items we should convert the analysis to represent the probability of a single item surviving for a specified period time. Assuming each item functions independently of each other we achieve this through calculating the 200<sup>th</sup> root of the survival probabilities for each root cause, as the time that the fleet first detects a fault is a minimum order statistic from a sample of 200. The survival probabilities for a single item are summarised in Table 4. It is clear that approximately 4% of the fleet will not survive the first time period but two third of the fleet will survive the first 4 time periods.

Table 4 Probability of an item surviving each time period by root cause and overall

<i>Time</i>	<i>Root Cause 1</i>	<i>Root Cause 4</i>	<i>Root Cause 8</i>	<i>Item</i>
1	0.9824	0.9813	0.9961	0.9612
2	0.9759	0.9747	0.9941	0.9456
3	0.9459	0.9091	0.9761	0.8383
4	0.9218	0.7839	0.9185	0.6637
5	0.0002	0.00002	0.0334	0.0000

#### *Negative Binomial Prior & Binomial Prior*

In order to assess the sensitivity of the results to the Poisson prior distribution, analysis is conducted with a Negative Binomial prior assuming the same mean and a variance 10 times greater than the variance of the Poisson, as well as using a Binomial prior distribution with the same mean but with a variance equal to a 10<sup>th</sup> the variance of the Poisson prior. The difference between these priors is shown in Table 5 where we record the difference in the expected number of items in a fleet of 200 that would be operating beyond the specified time. The results show that using the Negative Binomial prior will increase the expected number of items surviving although not greatly. With the exception of time period 4, the differences were less than 1 item. For the Binomial prior distribution the reverse occurs whereby the number of items expected to survive is slightly fewer than under the Poisson model.

Table 5 Arithmetic difference in expected number of items in a fleet of 200 surviving specified time periods

<i>Time Period</i>	<i>Difference Negative Binomial Prior</i>	<i>Difference Binomial Prior</i>
1	0.0629	-0.0064
2	0.1167	-0.0119
3	0.9907	-0.1040
4	4.5137	-0.5072
5	0.5637	-2.2E-08

## 5. Summary and Conclusions

A methodology has been presented for estimating the reliability of a variant design based upon the integration of historical data for the operational experience of the original design and expert engineering judgement to express the differences between the design variants through the identification of potential faults and associated likelihoods. The problem which we consider is one where few observed failures are recorded for operational items and hence we are challenged to find robust inference. An Empirical Bayes methodology for supporting statistical inference has been developed. The literature suggests that the estimates resulting from the EB methodology are more accurate than traditional statistical methods. The methodology is considered appropriate for the specified problem due to the multitude of possible root causes that may exist. This leads to the possibility of pooling data for accuracy through the construction of empirical priors, while adjusting the pooled estimate for each root cause separately to result in a unique distribution for each root cause.

The approach we develop is an improvement over using the raw data on each root cause because of the intrinsic smoothing performed on the data. Consider the raw probability estimates in Table 2 where seven of the eight root causes had no observations beyond the fourth time period. This creates a sharp finite support empirical distribution for the time to realise faults within these root causes, while two of the root causes would only permit faults to be realised within one time period. The usefulness of smoothing data to improve inference is supported in the literature.

It can be argued that the EB approach developed is an improvement over parametric modelling of the rate of occurrence of faults within operation because most models within the literature propose a smooth monotonically changing intensity function, while we propose a non-parametric model through the multinomial distribution. The illustrative example provides evidence of a bi-modal intensity function where time period 1 and 3 appear to be peaks within the realisation process.

The accuracy of the inference supported by this EB methodology increases as the stochastic behaviour of the root causes becomes more homogeneous. Hence the next stage in developing this methodology to develop data analysis techniques for homogenising the root causes.

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